

# An Improper Arithmetically Closed Borel Subalgebra of $\mathcal{P}(\omega)$ mod **FIN**

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## Abstract

We show the existence of a subalgebra  $\mathcal{A} \subseteq \mathcal{P}(\omega)$  that satisfies the following three conditions:

- $\mathcal{A}$  is Borel (when  $\mathcal{P}(\omega)$  is identified with  $2^\omega$ ).
- $\mathcal{A}$  is arithmetically closed (i.e.,  $\mathcal{A}$  is closed under the Turing jump, and Turing reducibility).
- The forcing notion  $(\mathcal{A}, \subseteq)$  modulo the ideal **FIN** of finite sets collapses the continuum to  $\aleph_0$ .

## 1. INTRODUCTION<sup>1</sup>

For a subalgebra  $\mathcal{A} \subseteq \mathcal{P}(\omega)$  let  $\mathbb{P}_{\mathcal{A}}$  be the partial order obtained by reducing  $(\mathcal{A}, \subseteq)$  modulo the ideal **FIN** of finite sets. Gitman [G-1] made an advance towards the *Scott set problem* by showing that, assuming the proper forcing axiom (PFA), if  $\mathcal{A}$  is arithmetically closed and  $\mathbb{P}_{\mathcal{A}}$  is a proper notion of forcing, then there is a model of Peano arithmetic whose standard system is

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$\mathcal{A}$ .<sup>2</sup> Gitman [G-2] also investigated proper posets of the form  $\mathbb{P}_{\mathcal{A}}$  and showed that the existence of proper uncountable arithmetically closed algebras  $\mathcal{A} \neq \mathcal{P}(\omega)$  is consistent with ZFC. These results naturally motivate the question whether there is an arithmetically closed  $\mathcal{A}$  for which  $\mathbb{P}_{\mathcal{A}}$  is not proper. This question was answered in the affirmative by Enayat [E, Theorem D], using a highly nonconstructive reasoning that establishes the existence of an arithmetically closed  $\mathcal{A}$  of power  $\aleph_1$  such that  $\mathbb{P}_{\mathcal{A}}$  collapses  $\aleph_1$  (and is therefore not proper). The nonconstructive feature of the proof prompted Question II(b) of [E], which asked whether  $\mathbb{P}_{\mathcal{A}}$  is a proper poset if  $\mathcal{A}$  is both arithmetically closed and *Borel* (when  $\mathcal{P}(\omega)$  is identified with the Cantor set).<sup>3</sup>

The main result of this paper, Theorem A below, provides a strong negative answer to the above question.

**Theorem A.** *There is an arithmetically closed Borel subalgebra  $\mathcal{A} \subseteq \mathcal{P}(\omega)$  such that  $\mathbb{P}_{\mathcal{A}}$  is equivalent to  $\text{LEVY}(\aleph_0, 2^{\aleph_0})$ .*

Theorem A is established in Section 3 using a rich toolkit from set theory and model theory. For this reason, Section 2 is devoted to the description of the machinery employed in the proof of Theorem A.

**Dedication.** We are honored to present this paper in a special issue that celebrates Ken Kunen's far reaching achievements.

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<sup>2</sup>The Scott set problem [KS, Question 1] asks whether every Scott set  $\mathcal{A}$  can be realized as the standard system of a model of Peano arithmetic (a subalgebra  $\mathcal{A} \subseteq \mathcal{P}(\omega)$  is a Scott set if  $\mathcal{A}$  is closed under Turing reducibility, and every infinite subtree of  ${}^{<\omega}2$  that is coded in  $\mathcal{A}$  has an infinite branch that is also coded in  $\mathcal{A}$ ). It is known that the answer to the Scott set problem is positive when  $|\mathcal{A}| \leq \aleph_1$ , and when  $\mathcal{A} = \mathcal{P}(\omega)$ . On the other hand, a subalgebra  $\mathcal{A} \subseteq \mathcal{P}(\omega)$  is arithmetically closed if  $\mathcal{A}$  is closed under (1) Turing jump and (2) Turing reducibility. Note that if  $\mathcal{A}$  is arithmetically closed, then  $\mathcal{A}$  is a Scott set, but not vice versa.

<sup>3</sup>Many of the other questions posed in [E] have by now been answered by Shelah; see [Sh-5] and [Sh-6].

## 2. PRELIMINARIES

### 2.1. FORCING

Given an infinite cardinal  $\kappa$ ,  $\text{LEVY}(\aleph_0, \kappa)$  is the usual partial order that collapses  $\kappa$  to  $\aleph_0$ , i.e.,  $\text{LEVY}(\aleph_0, \kappa) = (\langle \omega, \kappa, \subseteq \rangle)$ . The following result provides a structural characterization of  $\text{LEVY}(\aleph_0, \kappa)$ .<sup>4</sup>

**2.1.1. Theorem** (McAloon [Ko, Theorem 14.17]). *The following conditions are equivalent for a partial order  $\mathbb{P}$  of infinite cardinality  $\kappa$ .*

- (a)  $\mathbb{P}$  is equivalent<sup>5</sup> to  $\text{LEVY}(\aleph_0, \kappa)$ .
- (b)  $\mathbb{P}$  is  $(\aleph_0, \kappa)$ -nowhere distributive, i.e., there is a family  $\{I_n : n \in \omega\}$  of maximal antichains of  $\mathbb{P}$  such that for every  $p \in \mathbb{P}$ , there is some  $n < \omega$  such that there are  $\kappa$  elements of  $I_n$  that are compatible with  $p$ .

**2.1.2. Corollary** [J, Lemma 26.7]. *The following conditions are equivalent for a partial order  $\mathbb{P}$  of cardinality  $\kappa \geq \aleph_0$ .*

- (a)  $\mathbb{P}$  is equivalent to  $\text{LEVY}(\aleph_0, \kappa)$ .
- (b)  $\mathbb{V}^{\mathbb{P}} \models$  “there is a surjection  $f : \omega \rightarrow \kappa$ ”.

The next result shows that one can use standard techniques to build a subalgebra  $\mathcal{A} \subseteq \mathcal{P}(\omega)$  such that  $\mathbb{P}_{\mathcal{A}}$  is not proper.

**2.1.3. Proposition**<sup>6</sup>. *There is a family  $\mathcal{A} \subseteq \mathcal{P}(\omega)$  of cardinality  $\aleph_1$  such that  $\mathbb{P}_{\mathcal{A}}$  is equivalent to  $\text{LEVY}(\aleph_0, \aleph_1)$ .*

**Proof:** By a theorem of Parovičenko [Pa] (see also [Ko, Sections 5.28 and 5.29]) every Boolean algebra of cardinality  $\leq \aleph_1$  can be embedded into  $\mathcal{P}(\omega) \text{ mod FIN}$ . On the other hand,  $\text{LEVY}(\aleph_0, \aleph_1)$  can be densely embedded into a Boolean algebra of power  $\aleph_1$  since each  $s$  in  $\text{LEVY}(\aleph_0, \aleph_1)$  determines a basic clopen  $X_s$  set in  ${}^\omega\omega_1$ , and the Boolean algebra  $\mathbb{B}$  of clopen sets generated by the family  $\{X_s : s \in \text{LEVY}(\aleph_0, \aleph_1)\}$  is of size  $\aleph_1$ . So by Parovičenko’s theorem there is an embedding  $f$  of  $\mathbb{B}$  into  $\mathcal{P}(\omega) \text{ mod FIN}$ . Let  $\mathcal{A} := \{X \subseteq \omega : [X] \text{ is in the range of } f\}$ . Since  $\mathbb{P}_{\mathcal{A}}$  is isomorphic to  $\mathbb{B}$ , and  $\mathbb{B}$  is equivalent

<sup>4</sup>See [BS, Corollary 1.15] for a generalization of Theorem 2.1.1. that characterizes other collapsing algebras.

<sup>5</sup>For partial orders  $\mathbb{P}_1$  and  $\mathbb{P}_2$ ,  $\mathbb{P}_1$  is equivalent to  $\mathbb{P}_2$  if they yield the same generic extensions. This can be recast algebraically as the existence of an isomorphism between  $B(\overline{\mathbb{P}}_1)$  and  $B(\overline{\mathbb{P}}_2)$ , where  $\overline{\mathbb{P}}$  is the separative quotient of  $\mathbb{P}$ , and  $B(\overline{\mathbb{P}})$  is the complete Boolean algebra consisting of regular cuts of  $\overline{\mathbb{P}}$  [J, Theorem 14.10].

<sup>6</sup>Thanks to K.P. Hart and Ken Kunen for (independently) drawing our attention to this consequence of Parovičenko’s theorem.

to  $\text{LEVY}(\aleph_0, \aleph_1)$ ,  $\mathbb{P}_{\mathcal{A}}$  collapses  $\aleph_1$ . Therefore we may conclude that  $\mathbb{P}_{\mathcal{A}}$  is equivalent to  $\text{LEVY}(\aleph_0, \aleph_1)$  since  $\text{LEVY}(\aleph_0, \aleph_1)$  densely embeds into  $\mathbb{P}_{\mathcal{A}}$ .  $\square$

**2.1.4. Remark<sup>7</sup>.** Zapletal [Z, Lemma 2.3.1] used Woodin’s  $\Sigma_1^2$ -absoluteness theorem [L, Theorem 3.2.1] to show that in the presence of the continuum hypothesis and large cardinals (more precisely: a measurable Woodin cardinal), a projective partial order  $\mathbb{P}$  preserves  $\aleph_1$  iff  $\mathbb{P}$  is proper. Note that if  $\mathcal{A}$  is Borel, then  $\mathbb{P}_{\mathcal{A}}$  is projective.

## 2.2. INFINITE COMBINATORICS

**2.2.1. Definition.** Let  $\mathcal{A} \subseteq [\omega]^\omega := \{X \subseteq \omega : X \text{ is infinite}\}$ .

(a)  $\mathcal{A}$  is *almost disjoint* (AD) if the intersection of any two distinct members of  $\mathcal{A}$  is finite.

(b) An AD family  $\mathcal{A}$  is *maximal almost disjoint* (MAD) if  $\mathcal{A}$  has no proper extension to another AD family.

(c) A MAD family  $\mathcal{A}$  is *completely separable*<sup>8</sup> if for every set  $B \in [\omega]^\omega$  either there is some  $A \in \mathcal{A}$  such that  $A \subseteq B$  or  $B$  is a subset of the union of a finite subfamily of  $\mathcal{A}$ .

Note that if  $\mathcal{A}$  is a finite partition of  $\omega$ , then  $\mathcal{A}$  is completely separable. A routine diagonal argument, on the other hand, shows that any *infinite* MAD family  $\mathcal{A} \subseteq [\omega]^\omega$  must be uncountable; and indeed it is consistent with ZFC for a MAD family to have cardinality  $\aleph_1$  and  $2^{\aleph_0}$  to be arbitrarily large (e.g., by adding enough Cohen reals to a model of CH). However, if  $\mathcal{A}$  is an infinite completely separable MAD family, then the cardinality of  $\mathcal{A}$  must be  $2^{\aleph_0}$ . This follows from the well-known fact that if  $\mathcal{A}$  is completely separable, and  $B \subseteq \omega$  is not a subset of the union of a finite subfamily of  $\mathcal{A}$ , then  $\{A \in \mathcal{A} : A \subseteq B\}$  has cardinality continuum. *In particular, if  $\mathcal{A}$  is an infinite completely separable MAD family and  $B \subseteq \omega$ , then  $\{A \in \mathcal{A} : A \cap B \text{ is infinite}\}$  is either finite or has cardinality continuum.*

Hechler [H, Theorem 8.2, Lemma 9.2] showed that Martin’s axiom (MA) implies the existence of a completely separable family. A similar proof yields the following result.

<sup>7</sup>We owe this remark to Paul Larson.

<sup>8</sup>Completely separable families were first introduced in [H], and are referred to as “saturated families” in [GJS]. The question of the existence of an infinite completely separable MAD family in ZFC, posed by Erdős and Shelah [ES], remains open. Shelah [Sh-7] has recently shown that (1) the existence of such families can be established within  $\text{ZFC} + 2^{\aleph_0} < \aleph_\omega$ ; and (2) the nonexistence of such families has very high large cardinal strength. See also [HS] for further open questions and references.

**2.2.2. Theorem.** *The following statement (#) is provable within ZFC + MA.*

(#) *For every increasing sequence  $\bar{n} = \langle n_i : i < \omega \rangle$  with  $\lim_{i \in \omega} (n_{i+1} - n_i) = \infty$  there is a MAD family  $\mathcal{A} = \mathcal{A}_{\bar{n}}$  that satisfies the following two conditions:*

- (1)  $\mathcal{A} \subseteq \{u \subseteq \omega : \forall i \in \omega |u \cap [n_i, n_{i+1})| = 1\}$ , and
- (2) *If  $B \subseteq \omega$ , then  $\{A \in \mathcal{A} : A \cap B \text{ is infinite}\}$  is either finite or has cardinality  $2^{\aleph_0}$ .*

### 2.3. TREE INDISCERNIBLES

**2.3.1. Definition.** Suppose  $\mathcal{M}$  is a model with signature  $\tau_{\mathcal{M}}$ . An indexed family  $\{a_\eta : \eta \in {}^\omega 2\}$  of pairwise distinct elements of  $\mathcal{M}$  is said to be a family of *tree indiscernibles in  $\mathcal{M}$*  if for every  $\varphi(x_0, \dots, x_{m-1}) \in L_{\omega, \omega}(\tau_{\mathcal{M}})$ , there is some  $n_\varphi < \omega$ , such that for all natural numbers  $n > n_\varphi$  and all infinite sequences  $\eta_0, \dots, \eta_{m-1} \in {}^\omega 2$ ,  $\nu_0, \dots, \nu_{m-1} \in {}^\omega 2$  the following implication is true

$$\left( \bigwedge_{i < m} \eta_i \upharpoonright n = \nu_i \upharpoonright n \right) \wedge \left( \bigwedge_{i < j < m} \eta_i \upharpoonright n \neq \eta_j \upharpoonright n \right) \implies \\ \mathcal{M} \models \varphi[a_{\eta_0}, \dots, a_{\eta_{m-1}}] \leftrightarrow \varphi[a_{\nu_0}, \dots, a_{\nu_{m-1}}].$$

Tree indiscernibles were invented by Shelah ([Sh-1], [Sh-2]) to prove certain 2-cardinal theorems, including  $(\aleph_\omega, \aleph_0) \rightarrow (2^{\aleph_0}, \aleph_0)$ .<sup>9</sup> More recently, Shelah [Sh-4] further developed the machinery of tree indiscernibles in his work on Borel structures. In particular, he isolated a cardinal  $\lambda_{\omega_1}(\aleph_0)$  that satisfies the following three properties.

- $\lambda_{\omega_1}(\aleph_0) \leq \beth_{\omega_1}$  [Sh-4, Def. 1.1, Conclusion 1.8].
- $\lambda_{\omega_1}(\aleph_0)$  is preserved in c.c.c. extensions [Sh-4, Claim 1.10].

<sup>9</sup>Shelah also employed tree indiscernibles in his work on classification theory [Sh-3, VII, Sec.4] to show that for all  $\lambda \geq \max\{|T|, \aleph_1\}$   $T$  has  $2^\lambda$  nonisomorphic models of cardinality  $\lambda$  for every complete theory  $T$  that is not superstable ([Pi] includes an expository account). Tree indiscernibles were also discovered by Paris and Mills ([PM], [KS, Theorem 3.5.3]) in the context of nonstandard models of Peano arithmetic to show, e.g., the existence of model  $\mathcal{M}$  of PA with a nonstandard integer  $m$  in  $\mathcal{M}$  such that the set of  $\mathcal{M}$ -predecessors of  $m$  is externally countable but the set of  $\mathcal{M}$ -predecessors of  $2^m$  is of power  $2^{\aleph_0}$  (this result is also an immediate corollary of [Sh-1, Theorem 1]).

- If a sentence  $\psi \in \mathbf{L}_{\omega_1, \omega}$  has a model  $\mathcal{M}_0$  with  $|R^{\mathcal{M}_0}| \geq \lambda_{\omega_1}(\aleph_0)$  (where  $R$  is a distinguished unary predicate of  $\mathcal{M}_0$ ), then  $\psi$  has a Skolemized model  $\mathcal{M}$  that is generated by a family of tree indiscernibles in  $R^{\mathcal{M}}$  (in particular  $|R^{\mathcal{M}}| = 2^{\aleph_0}$ ) [Sh-4, Claim 2.1].

The above three facts immediately imply the following result. Note that  $\mathbf{L}_{\omega_1, \omega}$  might gain new sentences in the passage to a generic extension.

**2.3.2. Theorem.** *Suppose  $\mathbf{V}$  satisfies  $\aleph_{\omega_1} = \beth_{\omega_1}$  and  $\mathbb{P}$  is a c.c.c. notion of forcing. Then the following statement ( $\blacktriangle$ ) holds in  $\mathbf{V}^{\mathbb{P}}$ :*

( $\blacktriangle$ ) *If a sentence  $\psi$  of  $\mathbf{L}_{\omega_1, \omega}$  has a model  $\mathcal{M}_0$  with  $|R^{\mathcal{M}_0}| \geq \aleph_{\omega_1}$ , then there is a countable first order Skolemized theory  $T$  such that the signature  $\tau(T)$  of  $T$  extends the signature  $\tau(\psi)$  of  $\psi$ , and  $T + \psi$  has a model  $\mathcal{M}$  that is generated from a family of tree indiscernibles  $\{a_\eta : \eta \in {}^\omega 2\} \subseteq R^{\mathcal{M}}$ .*

The next result shows that for a given sentence  $\psi$  of  $\mathbf{L}_{\omega_1, \omega}$  the existence of a model of  $\psi$  that is generated by tree indiscernibles is absolute<sup>10</sup>.

**2.3.3. Theorem.** *For any sentence  $\psi$  of  $\mathbf{L}_{\omega_1, \omega}$  in  $\mathbf{V}$  the following statement ( $\spadesuit$ ) is absolute between  $\mathbf{V}$  and any generic extension  $\mathbf{V}^{\mathbb{P}}$ :*

( $\spadesuit$ ) *There is a Skolemized model  $\mathcal{M} \models \psi$  with a countable signature  $\tau(\mathcal{M}) \supseteq \tau(\psi)$  such that  $\mathcal{M}$  is generated from a family of tree indiscernibles  $\{a_\eta : \eta \in {}^\omega 2\} \subseteq R^{\mathcal{M}}$ .*

**Proof:** It is well known<sup>11</sup> that for any sentence  $\psi$  of  $\mathbf{L}_{\omega_1, \omega}$  with signature  $\tau(\psi)$  there is a countable Skolemized first order theory  $T_\psi$  in a countable signature  $\tau^+ \supseteq \tau(\psi)$  and a countable set  $\Gamma_\psi$  of 1-types of  $\tau^+$  such that (1) every model  $\mathcal{M}$  of  $\psi$  has an expansion to a model  $\mathcal{M}^+$  of  $T_\psi$  which omits the types in  $\Gamma_\psi$ , and (2) every model of  $T_\psi$  that omits the types in  $\Gamma_\psi$  satisfies  $\psi$ . Suppose  $\psi$  has a model  $\mathcal{M}$  generated from a family  $\{a_\eta : \eta \in {}^\omega 2\}$  of tree indiscernibles in  $\mathbf{V}^{\mathbb{P}}$ . Then in  $\mathbf{V}^{\mathbb{P}}$  we can form the multi-sorted structure  $(\mathcal{M}^+, \mathcal{N}, f)$ , where  $\mathcal{N}$  is the standard model for second order number theory  $(\omega, \mathcal{P}(\omega))$  (which is itself a two-sorted structure) and  $f : \mathcal{P}(\omega) \rightarrow R^{\mathcal{M}}$  by  $f(A) = a_{\chi_A}$  (where  $\chi_A$  is the characteristic function of  $A$ ). In particular, the signature  $\tau^*$  appropriate to  $(\mathcal{M}^+, \mathcal{N}, f)$  has a sort  $U_{\mathcal{M}}$  for the universe of  $\mathcal{M}^+$ , a sort  $U_{\mathcal{P}(\omega)}$  for  $\mathcal{P}(\omega)$ , and a sort  $U_\omega$  for  $\omega$ . Let  $\theta$  be the conjunction of the following sentences  $\theta_1, \dots, \theta_4$  of  $\mathbf{L}_{\omega_1, \omega}(\tau^*)$ . Note that  $\theta_4$  is the only finitary sentence in the list.

<sup>10</sup>This result is stated for generic extensions, but the proof shows that this absoluteness result is true for any two  $\omega$ -models  $\mathbf{V}$  and  $\mathbf{W}$  of  $\mathbf{ZF} + \mathbf{DC}$  with  $\mathcal{P}^{\mathbf{V}}(\omega) \subseteq \mathcal{P}^{\mathbf{W}}(\omega)$ .

<sup>11</sup>Cf. [Ke, Ch.11, Theorem 14] or [B, Theorem 6.18].

- $\theta_1$  expresses:  $\psi$  holds in  $U_M$ .
- $\theta_2$  expresses: the axioms of second order arithmetic<sup>12</sup> ( $Z_2$ ) hold in  $(U_{\mathcal{P}(\omega)}, U_\omega)$ .
- $\theta_3$  expresses:  $U_\omega$  is an  $\omega$ -model.
- $\theta_4$  expresses:  $f$  is an injection from  $\mathcal{P}(\omega)$  into  $R^M$ .

Consider the subset  $B$  of  $({}^\omega 2)^2$  that consists of elements of the form  $(r, s)$ , where  $r$  codes a countable model  $(\mathcal{M}_r^+, \mathcal{N}_r, f_r)$  of  $\theta$  such that  $\mathcal{M}_r^+$  omits the types in  $\Gamma_\psi$ , and  $s$  codes a function  $g_s : \omega \rightarrow \omega$  that witnesses the fact that the image of  $f_r$  forms a family of tree indiscernibles in the sense of  $\mathcal{N}_r$ , i.e.,  $g_s$  has the property that for every formula  $\varphi = \varphi(x_0, \dots, x_{m-1}) \in L_{\omega, \omega}(\tau_\psi)$ , if  $n > g_s(\ulcorner \varphi \urcorner)$ , then for all  $x_0, \dots, x_{m-1} \in U_{\mathcal{P}(\omega)}$ , and for all  $y_0, \dots, y_{m-1} \in U_{\mathcal{P}(\omega)}$  the following implication is true (in what follows,  $\varphi^{U_M}$  is the relativization of  $\varphi$  to  $U_M$ )

$$\left( \bigwedge_{i < m} \chi_{x_i} \upharpoonright n = \chi_{y_i} \upharpoonright n \right) \wedge \left( \bigwedge_{i < j < m} \chi_{x_i} \upharpoonright n \neq \chi_{x_j} \upharpoonright n \right) \rightarrow \varphi^{U_M}[f(x_0), \dots, f(x_{m-1})] \leftrightarrow \varphi^{U_M}[f(y_0), \dots, f(y_{m-1})].$$

It is easy to see that  $B$  is a Borel set with a Borel code  $c$  in  $\mathbf{V}$ . Also, by the downward Löwenheim-Skolem theorem for  $L_{\omega_1, \omega}$  sentences,

$$\mathbf{V}^{\mathbb{P}} \models \text{“the Borel set coded by } c \text{ is not empty”}.$$

On the other hand, the statement “the Borel set coded by  $c$  is empty” is provably equivalent (in  $\mathbf{ZF} + \mathbf{DC}$ ) to a  $\Pi_1^1$ -statement [J, Lemma 25.45] and therefore by Mostowski’s  $\Pi_1^1$ -absoluteness theorem [J, Theorem 25.4], the Borel set coded by  $c$  is nonempty in the real world  $\mathbf{V}$ . This shows that in  $\mathbf{V}$  there is a countable model  $(\mathcal{M}_0^+, \mathcal{N}_0, f_0)$  of  $\psi$ , and a function  $g_0 : \omega \rightarrow \omega$  that witnesses the fact that the image of  $f$  forms a family of tree indiscernibles in the sense of  $\mathcal{N}_0$  (in particular,  $\mathcal{N}_0$  is an  $\omega$ -model of second order arithmetic).

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<sup>12</sup>We just need a workable theory of finite and infinite sequences, so it is more than sufficient to use  $\mathbf{RCA}_0$  or  $\mathbf{ACA}_0$  instead of  $\mathbf{Z}_2$  here.

The countable model  $(\mathcal{M}_0^+, \mathcal{N}_0, f_0)$  and  $g_0$  together provide us with a blueprint  $\Sigma$  for producing a model of  $\psi$  of cardinality continuum that is generated by tree indiscernibles. To construct  $\Sigma$ , add new constants  $\{c_\eta : \eta \in {}^\omega 2\}$  to the vocabulary  $\tau^+$  of  $\mathcal{M}_0^+$ . Then  $\Sigma$  is defined as follows. Given  $\varphi(x_0, \dots, x_{m-1}) \in \mathbb{L}_{\omega, \omega}(\tau_{\mathcal{M}})$ , fix any  $n > g_s(\ulcorner \varphi \urcorner)$ , and find  $\nu_0 \cdots, \nu_{m-1} \in {}^\omega 2$  such that each  $\nu_i$  is coded in  $\mathcal{N}_0$  (i.e., there is some  $A_i$  in  $U_{\mathcal{P}(\omega)}$  of  $\mathcal{N}_0$  such that  $\chi_{A_i} = \nu_i$ ) and  $\eta_i \upharpoonright n = \nu_i \upharpoonright n$  for each  $i < m$ . Then put  $\varphi[c_{\eta_0}, \dots, c_{\eta_{m-1}}]$  or its negation in  $\Sigma$  depending on whether  $\mathcal{M}_0^+$  respectively satisfies  $\varphi[f_0(A_0), \dots, f_{m-1}(A_{m-1})]$  or its negation. Since  $\mathcal{M}_0^+$  is Skolemized,  $\Sigma$  uniquely determines an elementary extension  $\mathcal{M}_2^+$  of  $\mathcal{M}_0^+$  that is generated by tree indiscernibles. In order to arrange an elementary extension of  $\mathcal{M}_0^+$  that is generated by tree indiscernibles that also satisfies  $\psi$  we need to thin  $\mathcal{M}_2^+$  as follows. Since  $\mathcal{M}_0^+$  omits every type in  $\Gamma_\psi$  and  $g_0$  provides a witness to the tree indiscernibility of the range of  $f_0$ , we can easily construct a perfect subtree  $\Delta$  of  ${}^\omega 2$  such that the submodel  $\mathcal{M}_1^+$  of  $\mathcal{M}_2^+$  generated by  $\{c_\eta : \eta \in \Delta\}$  omits every type in  $\Gamma_\psi$ . Therefore  $\mathcal{M}_1^+$  is our desired model of  $\psi$  that is generated by tree indiscernibles.  $\square$

## 2.4. BOREL STRUCTURES

Recall that a model  $\mathcal{M}$  is said to be *totally Borel* if the universe of  $\mathcal{M}$  is a Borel subset of  $\mathbb{R}$ , and every subset of  $X$  that is parametrically definable in  $\mathcal{M}$  is a Borel set. It is known that every countable theory has an uncountable totally Borel model. This result was established by H. Friedman [St-1] and also (later, but independently) by Malitz-Mycielski-Reinhardt [MMR]. The following results are included for those readers favoring a shorter (albeit less self-contained) proof of Theorem A.

**2.4.1. Theorem** (Steinhorn [St-2]). *If  $\mathcal{M}$  is a model generated by tree indiscernibles, then  $\mathcal{M}$  is isomorphic to a totally Borel model.*

**2.4.2. Theorem** (Harrington-Shelah [HMS]) *No analytic linear order contains an uncountable well-ordered set. In particular, the cofinality of every Borel linear order with no last element is  $\aleph_0$ .*

## 3. PROOF OF THEOREM A

Before presenting the full technical details of the proof, let us describe a high-level summary of the three stages of the argument.

- **Stage 1 Outline.** Start with the constructible universe  $\mathbf{L}$  and a regular cardinal  $\kappa > (\aleph_{\omega_1})^{\mathbf{L}} = (\beth_{\omega_1})^{\mathbf{L}}$ . Then force  $\text{MA} + 2^{\aleph_0} = \kappa$  with the usual c.c.c. partial order  $\mathbb{Q}$  of cardinality  $\kappa$ . In  $\mathbf{L}^{\mathbb{Q}}$ , use Theorem 2.3.2 to get hold of an  $\omega$ -standard model  $\mathcal{M}'$  of  $\text{ZFC}^- + \text{MA}$  (where  $\text{ZFC}^-$  is  $\text{ZFC}$  without the powerset axiom) that is generated by tree indiscernibles.
- **Stage 2 Outline.** By Theorem 2.4.1  $\mathcal{M}'$  is a totally Borel model in  $\mathbf{L}^{\mathbb{Q}}$ . Combined with Theorem 2.3.3 this shows that there is also a totally Borel model  $\mathcal{M}$  in  $\mathbf{V}$  that shares the salient features of  $\mathcal{M}'$ . In particular,  $\mathcal{M}$  is an  $\omega$ -standard model of  $\text{ZFC}^-$  that satisfies  $\text{MA}$  and is generated by tree indiscernibles. The family  $\mathcal{A}$  of Theorem A is the set of reals of  $\mathcal{M}$ . This family  $\mathcal{A}$  is both Borel and arithmetically closed.
- **Stage 3 Outline.** By Theorem 2.4.2 every definable infinite linear order in  $\mathcal{M}$  with no last element has countable cofinality. This fact, when coupled with the veracity of  $\text{MA}$  in  $\mathcal{M}$ , will allow us to verify that  $\mathbb{P}_{\mathcal{A}}$  is  $(\aleph_0, 2^{\aleph_0})$ -nowhere distributive. By Theorem 2.1.1, this completes the proof of Theorem A.

We now proceed to flesh out the above outline.

**Stage 1.** Let  $\mu = (\aleph_{\omega_1})^{\mathbf{L}} = (\beth_{\omega_1})^{\mathbf{L}}$ , and fix a regular cardinal  $\kappa > \mu$ . By GCH in  $\mathbf{L}$ ,  $\kappa = \kappa^{<\kappa}$  holds in  $\mathbf{L}$ . Let  $\mathbb{Q}$  be the usual c.c.c. notion of forcing  $\text{MA} + 2^{\aleph_0} = \kappa$  [J, Theorem 16.13]. Let  $\mathcal{H}(\kappa^+)$  be the collection of sets whose transitive closure has cardinality at most  $\kappa$ . In the forcing extension  $\mathbf{L}^{\mathbb{Q}}$  let  $\mathcal{M}_0$  be an expansion of the structure  $(\mathcal{H}(\kappa^+), \in)$  by Skolem functions, a well-ordering of  $\mathcal{H}(\kappa^+)$ , and individual constants  $c_n$  and  $c_\omega$ , where  $c_n^{\mathcal{M}_0} = n$ , and  $c_\omega^{\mathcal{M}_0} = \omega$ . Let  $\tau = \tau_{\mathcal{M}_0}$  = the signature of  $\mathcal{M}_0$ . We may assume that  $\tau \in \mathbf{L}$  and  $\tau$  is countable in  $\mathbf{L}$ , but note that  $\text{Th}(\mathcal{M}_0)$  need not be in  $\mathbf{L}$ . Of course  $\mathcal{M}_0$  is a model of  $\text{ZFC}^- + \text{“}2^{\aleph_0}$  is the last cardinal” +  $\text{MA}$ .

Since  $\kappa > \mu$  we may invoke Theorem 2.3.2 to obtain a model  $\mathcal{M}'$  in  $\mathbf{L}^{\mathbb{Q}}$  that satisfies the following five conditions:

- (a')  $\mathcal{M}'$  is a model of  $\text{Th}(\mathcal{M}_0)$  with signature  $\tau$ . In particular  $\mathcal{M}'$  satisfies  $\text{ZFC}^- + \text{“}2^{\aleph_0}$  is the last cardinal” +  $\text{MA}$ .
- (b')  $\mathcal{M}'$  is an  $\omega$ -model, i.e.,  $\mathcal{M}'$  omits  $\{x \in c_\omega\} \cup \{x \neq c_n : n < \omega\}$ .
- (c') There is a family  $\{a'_\eta : \eta \in {}^\omega 2\}$  of tree indiscernibles in  $\mathcal{M}'$ .
- (d') For each  $\eta \in {}^\omega 2$ ,  $\mathcal{M}' \models \text{“}a'_\eta \subseteq c_\omega\text{”}$  (i.e., each  $a'_\eta$  is a real in the sense of  $\mathcal{M}'$ ).

(e')  $\mathcal{M}'$  is the Skolem hull of  $\{a'_\eta : \eta \in {}^\omega 2\}$ .

**Stage 2.** Let  $T = \{\varphi \in L_{\omega, \omega}(\tau) : 1 \Vdash_{\mathbb{Q}} \mathcal{M}' \models \varphi\}$ . Note that since  $\mathcal{M}'$  is actually a  $\mathbb{Q}$ -name,  $T \in \mathbf{L}$ . By Theorem 2.3.3 there is a  $\tau$ -model  $\mathcal{M}$  of  $T$  in  $\mathbf{V}$  and a family of tree indiscernibles  $\langle a_\eta : \eta \in {}^\omega 2 \rangle$  such that the following five conditions hold.

(a)  $\mathcal{M}$  is a model with signature  $\tau$  that satisfies  $\text{ZFC}^- + \text{“}2^{\aleph_0}$  is the last cardinal”} + MA.

(b)  $\mathcal{M}$  is an  $\omega$ -model.

(c) There is a family  $\{a_\eta : \eta \in {}^\omega 2\}$  of tree indiscernibles in  $\mathcal{M}$ .

(d) For each  $\eta \in {}^\omega 2$ ,  $\mathcal{M} \models \text{“}a_\eta \subseteq c_\omega\text{”}$ .

(e)  $\mathcal{M}$  is the Skolem hull of  $\{a_\eta : \eta \in {}^\omega 2\}$ .

We may assume that the model  $\mathcal{M}$  is in “reduced form”, i.e., the well-founded part of  $\mathcal{M}$  is transitive. In particular,  $\omega^{\mathcal{M}} = \omega$ , and if  $\mathcal{M} \models b \subseteq c_\omega$ , then  $b \in \mathcal{P}(\omega)$ . Let  $\mathcal{A} = \{b : \mathcal{M} \models b \subseteq c_\omega\}$ . Obviously  $\mathcal{A}$  is arithmetically closed<sup>13</sup>. By Theorem 2.4.1  $\mathcal{A}$  is also Borel. This fact can also be established directly as follows. For any  $\tau_{\mathcal{M}}$ -term  $\sigma = \sigma(x_0, \dots, x_{m-1})$ ,  $m < \omega$ ,  $n^* < \omega$ , and pairwise distinct  $\nu_0, \dots, \nu_{m-1} \in {}^{n^*} 2$ , let  $\bar{\nu} = \langle \nu_i : i < m \rangle$ , and consider the set  $\mathcal{A}_{\sigma, \bar{\nu}}$  defined as follows (in the formula below  $\triangleleft$  denotes the end extension relation among sequences):

$$\mathcal{A}_{\sigma, \bar{\nu}} := \{\omega \cap \sigma^{\mathcal{M}}(a_{\eta_0}, \dots, a_{\eta_{m-1}}) : \bigwedge_{i < m} \nu_i \triangleleft \eta_i \in {}^\omega 2\}.$$

It is sufficient to prove that  $\mathcal{A}_{\sigma, \bar{\nu}}$  is Borel for any  $(\sigma, \bar{\nu})$  since  $\mathcal{A}$  is the union of the countable family of sets of the form  $\mathcal{A}_{\sigma, \bar{\nu}}$ . We can find an increasing  $f : \omega \rightarrow \omega \setminus n^*$  and  $\langle g_n : n < \omega \rangle$  for which  $(\alpha)$ ,  $(\beta)$ , and  $(\gamma)$  hold.

( $\alpha$ )  $g_n$  is a function from  ${}^m (f(n)2)$  to  $\{0, 1\}$ .

( $\beta$ ) If  $\eta_0, \dots, \eta_{m-1} \in {}^\omega 2$  and  $\bigwedge_{i < m} \nu_i \triangleleft \eta_i \in {}^\omega 2$  and  $n < \omega$ , then (using tree indiscernibility)

$$n \in \sigma^{\mathcal{M}}(a_{\eta_0}, \dots, a_{\eta_{m-1}}) \Leftrightarrow g_n(\eta_0 \upharpoonright f(n), \dots, \eta_{m-1} \upharpoonright f(n)) = 1.$$

( $\gamma$ ) If  $A \subseteq \omega$ , then by König’s lemma,  $A \in \mathcal{A}_{\sigma, \bar{\nu}}$  iff for every  $n < \omega$  there are  $\rho_0, \dots, \rho_{m-1} \in f(n)2$  such that

<sup>13</sup>Indeed  $\mathcal{A}$  is even *hyperarithmetically* closed. This follows from the fact that any  $\omega$ -model of  $\Sigma_1^1\text{-AC}_0$  contains all hyperarithmetical sets [Si, Lemma VIII.4.15]; note that since  $\mathcal{M}$  satisfies the axiom of choice, the standard model of second order arithmetic in the sense of  $\mathcal{M}$  satisfies  $\Sigma_n^1\text{-AC}_0$  for all  $n < \omega$ .

$$k < n \Rightarrow (k \in A \Leftrightarrow g_k(\rho_0 \upharpoonright f(k), \dots, \rho_{m-1} \upharpoonright f(k)) = 1).$$

Note that  $(\gamma)$  shows that each  $\mathcal{A}_{\sigma, \bar{v}}$  is Borel.

**Stage 3:** By Theorems 2.4.1 and 2.4.2 every definable linear order  $(L, <_L)$  in  $\mathcal{M}$  with no last element has countable cofinality. Alternatively, one can argue directly as follows. Suppose to the contrary. Then for some regular uncountable cardinal  $\kappa$ , there is an increasing unbounded subset  $\{b_\alpha : \alpha < \kappa\}$  of  $(L, <_L)$ . Each  $b_\alpha$  can be written in  $\mathcal{M}$  as

$$b_\alpha = \sigma_\alpha(a_{\eta_0^\alpha}, \dots, a_{\eta_{n_\alpha-1}^\alpha}),$$

but without loss of generality, we may assume that (1)  $\sigma_\alpha = \sigma$ , (2)  $n_\alpha = n$ , (3)  $\{\{\eta_0^\alpha, \dots, \eta_{n-1}^\alpha\} : \alpha < \kappa\}$  forms a  $\Delta$ -system [J, Theorem 9.18], and (4)  $\eta_0^\alpha <_{\text{lex}} \eta_1^\alpha <_{\text{lex}} \dots$  (where  $<_{\text{lex}}$  denotes the lexicographic relation among binary sequences). In particular, we may assume that for some  $m < n$ ,

$$l < m \Rightarrow \eta_l^\alpha = \eta_l^0;$$

and

$$\eta_{l_1}^{\alpha_1} = \eta_{l_2}^{\alpha_2} \Rightarrow (l_1 = l_2 < m) \vee (\alpha_1, l_1) = (\alpha_2, l_2).$$

We can easily construct a countable  $Y \subseteq \kappa$  such that if  $\alpha < \kappa$  and  $k < \omega$ , then for some  $\beta \in Y$  we have

$$\bigwedge_{l < n} \eta_l^\alpha \upharpoonright k = \eta_l^\beta \upharpoonright k.$$

The proof would be complete once we verify that  $\{b_\beta : \beta \in Y\}$  is cofinal in  $(L, <_L)$ . Let  $\alpha < \kappa$ , and note that the concatenation of  $\langle \eta_l^\alpha : l < n \rangle$  and  $\langle \eta_l^{\alpha+1} : l \in [m, n) \rangle$  has no repetition. Choose  $k < \omega$  that satisfies the following condition  $(\nabla)$ :

$(\nabla)$  **If**  $\eta_l \in {}^\omega 2$  for  $l < n$ ,  $\nu_s \in {}^\omega 2$  for  $s \in [m, n)$ , and  $(\eta_l \upharpoonright k = \eta_l^\alpha \upharpoonright k) \wedge (\nu_s \upharpoonright k = \eta_s^{\alpha+1} \upharpoonright k)$ , **then**  $\mathcal{M}$  satisfies the following biconditional:

$$\begin{aligned} \sigma(\dots, a_{\eta_l^\alpha}, \dots) <_L \sigma(\dots, a_{\eta_l^{\alpha+1}}, \dots) &\leftrightarrow \\ \sigma(\dots, a_{\eta_l}, \dots) <_L \sigma(a_{\eta_0}, \dots, a_{\eta_{m-1}}, a_{\nu_m}, \dots, a_{\nu_{n-1}}). \end{aligned}$$

Lastly, choose  $\beta \in Y$  such that

$$\bigwedge_{l < n} \eta_l^\beta \upharpoonright k = \eta_l^{\alpha+1} \upharpoonright k.$$

Hence  $(b_\alpha <_L b_{\alpha+1}) \Leftrightarrow (b_\alpha <_L b_\beta)$ , which shows that  $\{b_\beta : \beta \in Y\}$  is cofinal in  $(L, <_L)$  and concludes the proof.

□ (Claim 3.1)

We now complete the proof of Theorem A by showing that  $\mathbb{P}_A$  is equivalent to  $\text{LEVY}(\aleph_0, 2^{\aleph_0})$ . By Theorem 2.1.1, it suffices to establish the following claim.

**Claim 3.2.** *There is a family  $\{I_n : n \in \omega\}$  of maximal antichains in  $\mathbb{P}_A$  such that for every  $p \in \mathbb{P}_A$  there is some  $n < \omega$  such that  $\{q \in I_n : p \text{ and } q \text{ are compatible}\}$  has cardinality  $2^{\aleph_0}$ .*

**Proof:** Recall that MA holds in  $\mathcal{M}$  (see condition (a) of Stage (2)). Hence  $\mathcal{M}$  satisfies “there is a  $2^{\aleph_0}$ -scale  $\{f_\alpha : \alpha < 2^{\aleph_0}\}$  in  $({}^\omega\omega, <_*)$ ” [J, Corollary 16.25]. In other words,  $\mathcal{M}$  satisfies

$$\forall g : \omega \rightarrow \omega \exists \alpha < 2^{\aleph_0} \text{ such that } g <_* f_\alpha \text{ (i.e., } g(n) < f_\alpha(n) \text{ for sufficiently large } n), \text{ and } f_\alpha <_* f_\beta \text{ whenever } \alpha < \beta < 2^{\aleph_0}.$$

Therefore, using Claim 3.1 we may fix a *countable* family of functions  $F = \{f_n : n \in \omega\} \subseteq {}^\omega\omega \cap \mathcal{M}$  such that for every  $g \in {}^\omega\omega \cap \mathcal{M}$  there is some  $f_n \in F$  such that  $g <_* f_n$ . Of course we may assume that  $f_n$  is an increasing function for each  $n \in \omega$ . For each  $f_n \in F$ , let  $\overline{f_n} \in \mathcal{M}$  be an auxiliary function defined by  $\overline{f_n}(0) = f_n(0)$  and

$$\forall i \in \omega \quad \overline{f_n}(i+1) = i + f_n(\overline{f_n}(i) + 2).$$

Since  $\mathcal{M}$  satisfies (#), and  $\lim_{n \in \omega} \overline{f_n}(i+1) - \overline{f_n}(i) = \infty$ , there is some family  $I_n \in \mathcal{M}$  with  $I_n \subseteq [\omega]^\omega$  such that for all  $A \in I_n$  and for all  $i < \omega$

$$|A \cap [\overline{f_n}(i), \overline{f_n}(i+1))| = 1,$$

and for each  $B \in \mathcal{P}(\omega) \cap \mathcal{M}$ ,  $\mathcal{M}$  satisfies

$$\{A \in \mathcal{A} : A \cap B \text{ is infinite}\} \text{ is either finite or has cardinality continuum.}$$

We now verify that  $\langle I_n : n < \omega \rangle$  exemplifies condition (b) of Theorem 2.1.1. Given a condition  $p = [B] \in \mathbb{P}_{\mathcal{A}}$ , we may assume that  $B$  is infinite. It is routine to construct a strictly increasing function  $g \in {}^\omega\omega \cap \mathcal{M}$  by recursion such that  $g(0) = 0$ , and

$$(1) \quad \forall k \quad |B \cap [g(k), g(k+1))| \geq g(k).$$

Choose  $f_n \in F$  and  $i_0 \in \omega$  such that  $g(i) < f_n(i)$  for all  $i \geq i_0$ , and let

$$Y := \{i : \exists k (\overline{f_n}(i) < g(k) < g(k+1) < \overline{f_n}(i+1))\}.$$

We wish to show that  $i \in Y$  for all  $i \geq i_0$  (the fact that  $Y$  is infinite will come handy below). To this end, suppose  $i \geq i_0$ . Since  $\langle g(s) : s < \omega \rangle$  is a strictly increasing sequence, we can find  $k < \omega$  such that  $k$  is the first  $s$  such that  $\overline{f_n}(i) < g(s)$ . Hence  $g(k-1) \leq \overline{f_n}(i)$ , which in turn implies that  $k-1 \leq \overline{f_n}(i)$  (since  $g$  is strictly increasing), and therefore

$$(2) \quad k+1 \leq \overline{f_n}(i) + 2.$$

Using the strictly increasing feature of  $g$  one more time, (2) yields

$$(3) \quad g(k+1) \leq g(\overline{f_n}(i) + 2).$$

On the other hand, since  $\overline{f_n}(i) + 2 \geq i \geq i_0$ , and  $f_n$  dominates  $g$  for  $i \geq i_0$

$$(4) \quad g(\overline{f_n}(i) + 2) < f_n(\overline{f_n}(i) + 2) < i + f_n(\overline{f_n}(i) + 2) = \overline{f_n}(i+1).$$

By putting (3) and (4) together, we obtain  $g(k+1) < \overline{f_n}(i+1)$ . This shows that  $Y$  includes every  $i \geq i_0$ .

Now let  $\mathcal{F}_B := \{A \in I_n : A \cap B \text{ is infinite}\}$ , and note that  $\mathcal{F}_B \in \mathcal{M}$ . Thanks to (#)  $\mathcal{M}$  satisfies “ $\mathcal{F}_B$  is finite or has cardinality continuum”. But  $\mathcal{F}_B$  cannot be finite, since each  $A \in \mathcal{F}_B$  has only one element in each interval  $[\overline{f_n}(i), \overline{f_n}(i+1))$ , whereas  $B$  has more than  $\overline{f_n}(i)$  members for infinitely many values of  $i$ , thanks to (1) and the fact that  $Y$  is infinite. Hence  $\mathcal{F}_B$  has cardinality  $2^{\aleph_0}$  in the sense of  $\mathcal{M}$ , and therefore in the real world as well, since  $\mathcal{M}$  has continuum-many reals.

□ (Claim 3.2).

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## References

- [BS] B. Balcar and P. Simon, *Disjoint Refinement*, in **Handbook of Boolean Algebras**. vol. 2, Edited by J. Donald Monk and Robert Bonnet, North-Holland Publishing Co., Amsterdam, 1989.
- [B] J. T. Baldwin, **Categoricity**, University Lecture Series, vol. 50, American Mathematical Society, Providence, RI, 2009.
- [E] A. Enayat, *A standard model of arithmetic with no conservative elementary end extension*, **Annals of Pure and Applied Logic**, vol. 156, Issues 2-3 (2008), pp. 308-318.
- [ES] P. Erdős and S. Shelah, *Separability properties of almost disjoint families of sets*, **Israel J. Math.** vol.12 (1972), pp. 207-214.
- [G-1] V. Gitman, *Can the proper forcing axiom give new solutions to the Scott set problem?* **Journal of Symbolic Logic**, vol. 73 (2008), pp. 845-860.
- [G-2] -----, *Proper and piecewise proper families of reals*, **Mathematical Logic Quarterly**, vol. 55 (2009), pp. 542-550.
- [GJS] M. Goldstern, H. Judah, and S. Shelah, *Saturated families*, **Proc. Amer. Math. Soc.** vol. 111 (1991), no. 4, 1095–1104.
- [HMS] L. Harrington, D. Marker, and S. Shelah, *Borel orderings*, **Trans. Amer. Math. Soc.** vol. 310 (1988), no. 1, 293–302.
- [H] S. H. Hechler, *Classifying almost disjoint families with applications to  $\beta N - N$* , **Israel J. Math.**, vol. 10 (1971), pp. 413-432.
- [HS] M. Hrušák and P. Simon, *Completely separable MAD families*, in **Open Problems in Topology II** (edited by Elliott Pearl), Elsevier B. V., Amsterdam, 2007, pp.179-184.
- [J] T. Jech, **Set Theory**, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.
- [Ke] H. J. Keisler, **Model Theory for Infinitary Logic**, North-Holland, Amsterdam (1971).
- [Ko] S. Koppelberg, *Algebraic Theory*, in vol. 1 of **Handbook of Boolean Algebras**, Edited by J. Donald Monk and Robert Bonnet, North-Holland Publishing Co, Amsterdam, 1989.

- [KS] R. Kossak and J.H. Schmerl, **The Structure of Models of Peano Arithmetic**, Oxford Logic Guides, vol. 50, Oxford Science Publications, The Clarendon Press, Oxford University Press, Oxford, 2006.
- [L] Paul Larson, **The Stationary Tower** (Notes on a course by W. Hugh Woodin), University Lecture Series, 32, American Mathematical Society, Providence, RI, 2004.
- [MMR] J. Malitz, J. Mycielski & W. Reinhardt, *The Axiom of Choice, the Löwenheim-Skolem theorem and Borel models*, **Fundamenta Mathematicae** vol. 137 (1991), pp. 53-58. [Erratum: Fund. Math. vol. 140 (1992), pp.197.]
- [PM] J. Paris and G. Mills, *Closure properties of countable nonstandard integers*. **Fundamenta Mathematicae** vol. 103 (1979), no. 3, pp. 205–215.
- [Pa] I. I. Parovičenko, *A universal bicomact of weight  $\aleph$* , **Soviet Mathematics Doklady** 4 (1963), pp. 592 (Russian original: Ob odnom universalnom bikompakte vesa  $\aleph$ , Doklady Akademii Nauk SSSR 150 (1963) pp. 36-39).
- [Pi] A. Pillay, *Number of models of theories with many types*, **Study Group on Stable Theories** (Bruno Poizat, ed.), Second year: 1978/79 (French), Exp. No. 9, 10 pp., Secrétariat Math., Paris, 1981.
- [Sh-1] S. Shelah, *A two-cardinal theorem*, **Proc. Amer. Math. Soc.** vol. 48 (1975), pp. 207–213.
- [Sh-2] -----, *A two-cardinal theorem and a combinatorial theorem*, **Proc. Amer. Math. Soc.** 62 (1977). pp. 134–136.
- [Sh-3] -----, **Classification theory and the Number of Nonisomorphic Models** (second edition), Studies in Logic and the Foundations of Mathematics, vol. 92. North-Holland Publishing Co., Amsterdam, 1990.
- [Sh-4] -----, *Borel sets with large squares*, **Fundamenta Mathematicae** vol. 159 (1999), no. 1, pp. 1–50.
- [Sh-5] -----, *Models of expansions of  $\mathbb{N}$  with no end extensions* (2010), arXiv:0808.2960.

- [Sh-6] \_\_\_\_\_, *Models of PA: Standard systems without minimal ultrafilters* (2009), arXiv:0901.1499.
- [Sh-7] \_\_\_\_\_, *MAD Families and SANE Player* (2009), arXiv:0904.0816.
- [Si] S. Simpson, **Subsystems of second order arithmetic**, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999.
- [St-1] C. Steinhorn, *Borel structures and measure and category logics*, in **Model-theoretic logics**, Perspect. Math. Logic, Springer-Verlag, New York, 1985, pp. 579-596.
- [St-2] \_\_\_\_\_, *Borel structures for first-order and extended logics*, in **Harvey Friedman's research on the foundations of mathematics**, edited by L. A. Harrington, M. D. Morley, A. Scedrov and S. G. Simpson, Studies in Logic and the Foundations of Mathematics, vol. 117. North-Holland Publishing Co., Amsterdam, 1985, pp. 161-178.
- [Z] J. Zapletal, **Descriptive set theory and definable forcing**, Mem. Amer. Math. Soc. 167 (2004), no. 793.

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